

Geometric invariance of mass-like asymptotic invariants

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Abstract

We study coordinate-invariance of some asymptotic invariants such as the ADM mass or the Chruściel-Herzlich momentum, given by an integral over a “boundary at infinity”. When changing the coordinates at infinity, some terms in the change of integrand do not decay fast enough to have a vanishing integral at infinity; but they may be gathered in a divergence, thus having vanishing integral over any closed hypersurface. This fact could only be checked after direct calculation (and was called a “curious cancellation”). We give a conceptual explanation thereof.

1 Introduction

General Relativity has introduced a new kind of geometric invariants that depend on the geometry “at infinity” of a non compact Riemannian manifold. Given such a (M, g) , and a reference Riemannian metric g_0 to which g is asymptotic, these invariants are formally defined as an integral over the “boundary at infinity” of M of a field of 1-forms $\mathbb{U}(g, g_0)$:

$$m(g, g_0) := \oint_{S_\infty} \mathbb{U}(g, g_0)(\nu) dS. \quad (1)$$

The integral over S_∞ is understood as a limit

$$\lim_{r \rightarrow \infty} \oint_{S_r} \mathbb{U}(g, g_0)(\nu) dS$$

where $(S_r)_r$ is a family of closed hypersurfaces enclosing the whole of M when $r \rightarrow \infty$, and ν and dS respectively are the outer unit normal and induced volume measure of S_r with respect to g_0 . In fact, g and g_0 could encode other geometric data, for example first and second fundamental forms of a space-like hypersurface in a space-time. The ADM and Abbott-Deser energy-momentum [3], [1], mathematically studied in terms of Cauchy data in [6], [4], [8], [10], belong to that category. We shall call such invariants *total charges*.

In general, many mutually isometric but distinct g_0 ’s exist and are asymptotic to the given g , for instance pulled-back Ψ^*g_0 when Ψ is any diffeomorphism of M suitably asymptotic to the identity. However, in the examples mentioned above, with appropriate decay conditions, it is proven ([6], [4], [8], [10]) that the resulting $m(g, g_0)$ does not depend on the particular chosen g_0 . The proof relies on an algebraic fact that we recall below, in the simply stated case of the ADM mass of an asymptotically flat manifold.

The Riemannian manifold (M, g) is said to be asymptotically flat when there exists a set of coordinates $(x^i)_i$ defined outside a compact subset of M , in which the metric coefficients satisfy $g_{ij} = \delta_{ij} + e_{ij}$, where $e_{ij} = O(|x|^{-\tau})$ and similar decay holds for $\partial_k e_{ij}$. The metric g is then asymptotic to the flat metric $g_0 := \sum dx^i{}^2$. One defines the 1-form

$$\begin{aligned} \mathbb{U}_i(g, g_0) &= \sum_j \partial_j g_{ij} - \partial_i g_{jj} \\ \text{i.e. } \mathbb{U}(g, g_0) &:= \operatorname{div}_{g_0} g - d(\operatorname{tr}_{g_0} g) \end{aligned}$$

and the ADM mass of g , *a priori* with respect to this particular chart at infinity, is given by formula (1).

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Let \hat{x}^i be other coordinates in which g is asymptotically flat. It is an intuitive—but non trivial—theorem that, maybe after rotating and translating the \hat{x}^i 's, one has $\hat{x}^i - x^i =: v^i = O(|x|^{1-\tau})$, and similar decay holds for two derivatives of v^i . An easy calculation then shows that the 1-forms \mathbb{U} defining the ADM mass in the two coordinate sets respectively are related by

$$\mathbb{U}(g, g_0) - \mathbb{U}(g, \hat{g}_0) = \sum_{i,j} \partial_i (\partial_j v^i - \partial_i v^j) dx^j + O(|x|^{-2\tau-1}).$$

Provided $\tau > \frac{n-2}{2}$, the decay of the last term is faster than the critical rate r^{1-n} (the volume of large coordinates spheres), so this term does not contribute to the limit of the integrals over large spheres. The first term in the right-hand side, however, decays slower than the critical rate. Invariance of the ADM mass comes from the fact that it happens to be the divergence of an alternating 2-form, so that its integral over any closed hypersurface vanishes. But this divergence curiously only appears in a direct calculation, knowing the explicit formula for $\mathbb{U}(g, g_0) - \mathbb{U}(g, \hat{g}_0)$, so that it was for example called a “curious cancellation” by R. Bartnik [4].

In the asymptotically hyperbolic setting, the definition of the Chruściel-Herzlich mass [8] is more elaborate and uses an auxiliary function in the integrand \mathbb{U} . Yet its invariance also relies on the appearance of a divergence gathering the terms above the critical decay rate, and still only in a direct calculation.

We give in this note a conceptual explanation to that “curious cancellation”. Simplifying a little, we follow the three following steps:

1. We give a general construction of a charge integrand \mathbb{U} and the related total charges,
2. We obtain a general expression for the difference of charge integrands computed in two different charts at infinity, modulo fast-decaying terms,
3. We prove that this expression is the divergence of a field of alternating 2-forms.

Point 1 follows M. Herzlich [11, §3.2]—see also [9] and [12]. In fact Points 2 and 3 answer Question 3.5 in the first reference. Both appear to be very simple. It must be noticed however that in Point 2 we pick up an expression among many other possible ones, all equal modulo fast decaying terms. To find the one appropriate for Point 3, one needs to guess in advance the phenomenon that occurs there, despite it may not be properly formulated without knowing the accurate formula of Point 2. The interest of the construction of Point 1 is then justified *a posteriori* only.

Our study is unrelated to Hamiltonian or Lagrangian formalism, see for example [15], which builds uniquely defined Hamiltonians, but only in restriction to phase space, *i.e.* geometric data that satisfy some constraints. (Moreover the dependence on the “appropriate decay conditions”, as usually named in the literature, is not very clear at a formal level.) Our construction is closer in spirit to that of [2], but different, in that we are not interested in conservation laws, but only in geometric invariants that are independent, to a certain extent, of the background. One may check indeed that the formulae given in [2] are not those obtained following the construction presented here.¹

The outline of the article is the following. In Section 2, we give the construction of the charge integrands \mathbb{U} considered in this paper, see Definition 2.1, and we use it to define a total charge in Definition 2.5. In Section 3 we prove the main result, Theorem 3.3, according to which the total charge is invariant under a diffeomorphism suitably asymptotic to the identity at infinity. It covers Points 2 and 3 above, which correspond respectively to formula (4) and the Cancellation Lemma page 6. The meaning of “suitably asymptotic to the identity” is quite technical to state accurately. Since it is not the main interest of this paper, the precise discussion is postponed to the appendix A. Before that, we show in Section 4 how our reasoning apply to already known invariants.

2 Total charge

Let M be a non compact differential manifold without boundary. The geometric data we consider here are sections $h = (g, k)$ of a bundle $H = \mathcal{M} \times_M E$ over M , where \mathcal{M} is the bundle of metrics and

¹A general result about gauge invariance seems to be claimed in [2], but unfortunately does not appear in the literature, even as a preprint.

E a natural tensor bundle. The role of local charge density will be played by a natural tensor-valued differential operator

$$\Phi : \Gamma(H) \longrightarrow \Gamma(F)$$

where F is a natural tensor bundle over M , and the letter Γ denotes the space of C^∞ sections. This operator needs not be linear, but we require invariance under diffeomorphisms: *i.e.* for all sections h of H and all diffeomorphisms Ψ of M ,

$$\Psi^*(\Phi(h)) = \Phi(\Psi^*h).$$

For clarity we will suppose that the background data are given on an other manifold M_0 . Let us write \mathcal{M}_0 , E_0 , and F_0 for the bundles over M_0 that correspond to \mathcal{M} , E , and F respectively. We denote by $h_0 = (g_0, k_0)$ a section of $H_0 := \mathcal{M}_0 \times_{M_0} E_0$ that will be used as reference. Because of naturality, Φ is defined on M_0 ; we set $\Phi_0 := \Phi(h_0)$. We also introduce the dot product $\langle \cdot, \cdot \rangle_0$ and norm $|\cdot|_0$ induced by g_0 on natural tensor bundles over M_0 .

We now define \mathbb{U} .

Definition 2.1. Let $D\Phi_0$ be the linearization of Φ at $h_0 = (g_0, k_0)$ and $D\Phi_0^*$ be its formal adjoint with respect to g_0 .

For a C^∞ section V of F_0 and a C^∞ section η of $S^2M_0 \times_{M_0} E_0$, the charge integrand $\mathbb{U}(V, \eta)$ is the 1-form appearing in the following integration-by-part formula:

$$\langle V, D\Phi_0(\eta) \rangle_0 = \text{div}_0 \mathbb{U}(V, \eta) + \langle D\Phi_0^*V, \eta \rangle_0.$$

Here div_0 is the g_0 -divergence operator: if ∇ is the Levi-Civita connection of g_0 and α is a form field, $\text{div}_0 \alpha = \nabla^i \alpha_i$.

Remark 2.2. The operator \mathbb{U} is a differential operator, linear and of order 1 less than Φ in each of its arguments.

To use \mathbb{U} to define asymptotic invariants, we are interested in the situation where the outside of a compact subset of M is diffeomorphic to the outside of a compact subset of M_0 , *i.e.* when the following definition is not empty:

Definition 2.3. A *diffeomorphism at infinity* is a diffeomorphism

$$\Psi : M_0 - K_0 \longrightarrow M - K$$

where K_0 and K are compact subsets of respectively M_0 and M .

Let h be a section of H , and set $e := \Psi^*h - h_0$ outside K_0 . We will require e to tend to 0 at infinity (in a sense to be made precise below). This justifies the use of the following Taylor formula, which however always makes sense: $\Phi(\Psi^*h) - \Phi_0 = D\Phi_0(e) + Q(e)$, where $Q(e)$ is the quadratic and higher order remainder. Contracting with a test-section V of F_0 to get a numerical value, we obtain:

$$\begin{aligned} \langle V, \Phi(\Psi^*h) - \Phi_0 \rangle_0 &= \langle V, D\Phi_0(e) \rangle_0 + Q(V, e) \\ &= \text{div}_0 \mathbb{U}(V, e) + \langle D\Phi_0^*(V), e \rangle_0 + Q(V, e) \end{aligned} \quad (2)$$

where $Q(V, e) := \langle V, Q(e) \rangle_0$ for short, and Definition 2.1 has been used. Let us introduce

$$\mathcal{N}_0 := \{V \in \Gamma(F_0) | D\Phi_0^*V = 0\}.$$

When $V \in \mathcal{N}_0$, the right-hand side of (2) contains only a divergence plus terms of quadratic and higher order in e . Thus we are interested in the following class of data h :

Definition 2.4. A section h of H is said to have *well-defined total charge* with respect to a diffeomorphism at infinity Ψ and to $V \in \mathcal{N}_0$ (respectively to a subspace $\mathcal{N}'_0 \subset \mathcal{N}_0$) when:

1. $\langle V, \Phi(\Psi^*h) - \Phi_0 \rangle_0$ is integrable (with respect to the volume density induced by g_0),
 2. writing $e = \Psi^*h - h_0$, $Q(V, e)$ is integrable
- (respectively when 1 and 2 hold for all $V \in \mathcal{N}'_0$).

For such h 's one imitates the classical definitions:

Definition 2.5. Let h have well-defined total charge with respect to some diffeomorphism at infinity Ψ and some $V \in \mathcal{N}_0$.

Let $(B_k)_{k \in \mathbb{N}}$ be an non-decreasing exhaustion of M_0 such that each B_k has smooth compact boundary S_k . One defines the *total charge* as the following limit

$$m(h, \Psi, V) := \lim_{k \rightarrow \infty} \oint_{S_k} \mathbb{U}(V, \Psi^*h - h_0)(\nu) dS$$

where ν and dS are the outer normal and surface measure of S_k with respect to g_0 .

The limit is finite and independent of the chosen exhaustion (B_k) . Let us recall the classical proof of this fact.

Proof. When $V \in \mathcal{N}_0$, Equation (2) becomes

$$\operatorname{div}_0 \mathbb{U}(V, e) = \langle V, \Phi(\Psi^*h) - \Phi_0 \rangle_0 - Q(V, e).$$

There exists k_1 such that for $k \geq k_1$, $M_0 - B_k$ is included in the domain $M_0 - K_0$ of Ψ . Integrating over $B_k - B_{k_1}$ with respect to the volume element dvol_0 of g_0 :

$$\oint_{S_k} \mathbb{U}(V, e)(\nu) dS = \oint_{S_{k_1}} \mathbb{U}(V, e)(\nu) dS + \int_{B_k - B_{k_1}} \left[\langle V, \Phi(\Psi^*h) - \Phi_0 \rangle_0 - Q(V, e) \right] \operatorname{dvol}_0.$$

Definition 2.4 insures that the right-hand side has finite limit when $k \rightarrow \infty$. Moreover the formula shows that the right-hand side is independent of k_1 ; in fact B_{k_1} and S_{k_1} there could be replaced by any B and $S = \partial B$ respectively, without changing its value (provided B is large enough for Ψ to be defined on $M_0 - B$). Thus the limit

$$\oint_{S_{k_1}} \mathbb{U}(V, e)(\nu) dS + \int_{M_0 - B_{k_1}} \left[\langle V, \Phi(\Psi^*h) - \Phi_0 \rangle_0 - Q(V, e) \right] \operatorname{dvol}_0$$

is independent of the exhaustion (B_k) . □

Remark 2.6. Let G_0 be the group of diffeomorphisms of M_0 fixing h_0 . It acts on \mathcal{N}_0 by pull-back. The total charge is of particular interest when h satisfies Definition 2.4 with respect to some Ψ and to a subspace $\mathcal{N}'_0 \subset \mathcal{N}_0$ invariant under G_0 . Indeed one straightforwardly checks (due to the independence with respect to the exhaustion B_k) that the total charge functional m is then a G_0 -equivariant linear form on \mathcal{N}'_0 , in the sense that for all $A \in G_0$ and $V \in \mathcal{N}'_0$

$$m(h, \Psi, A^*V) = m(h, \Psi \circ A^{-1}, V).$$

Remark 2.7. In general, \mathcal{N}_0 may contain only the zero section of F_0 , making Definitions 2.4 and 2.5 trivial. There are however many examples where it does not, see Section 4.

3 Geometric invariance

We compare the total charges given by two diffeomorphisms at infinity

$$\begin{array}{ccc} & (M, h) & \\ \Psi_1 \nearrow & & \nwarrow \Psi_2 \\ (M_0, h_0) & & (M_0, h_0) \end{array}$$

such that h has well-defined total charge with respect to some $V \in \mathcal{N}_0$ and to both Ψ_1 and Ψ_2 . We assume that $\Psi := \Psi_1^{-1} \circ \Psi_2$ tends to the identity at infinity, in the following sense.

Let $\exp : TM_0 \rightarrow M_0$ be the exponential map of g_0 . For a section ζ of TM_0 , let us write $\exp \circ \zeta : x \mapsto \exp_x(\zeta(x))$. Since $\exp \circ 0 = \operatorname{Id}_{M_0}$, there is a C^1 neighborhood \mathcal{U} of the zero section of TM_0 such that for all $\zeta \in \mathcal{U}$, $\exp \circ \zeta$ is a diffeomorphism such that $1/4g_0 \leq (\exp \circ \zeta)^*g_0 \leq 4g_0$ (see Proposition A.1-part 1).

Definition 3.1. A diffeomorphism at infinity $\Psi : M_0 - K_1 \longrightarrow M_0 - K_2$ is said to be *asymptotic to the identity* when outside a compact subset one has

$$\Psi(x) = \exp_x(\zeta(x)),$$

with $\zeta \in \mathcal{U}$ a smooth vector field.

Remark 3.2. We shall prove that the total charge is invariant under a change of diffeomorphism at infinity that is asymptotic to the identity. This is a restrictive assumption. Notice however that there are many cases of interest where any Ψ such that Ψ^*h_0 satisfy appropriate decay towards h_0 may be written $\Psi_0 \circ A$, where A fixes h_0 and Ψ_0 is asymptotic to the identity in the sense given above. In this case, the total charge becomes a genuine linear form on \mathcal{N}_0 , equivariant under the group of automorphisms of h_0 . We call this feature *asymptotic rigidity* of the background datum.

Examples of asymptotically rigid backgrounds are the Euclidean space \mathbb{R}^n (where h_0 is the canonical flat metric) [6],[4], the hyperbolic space [10], [8] or more generally a symmetric rank-1 space of non-compact type [11], or, in a more complicated way, the Minkowski space at spatial infinity [7]. Asymptotic rigidity is however not general: for example it is well known not to hold at null infinity in the Minkowski space when the decay conditions allow gravitational radiation.

Let us write $h_1 := \Psi_1^*h$, $e_1 := h_1 - h_0$ and $e_2 := \Psi_2^*h - h_0 = \Psi^*h_1 - h_0$. One computes

$$\begin{aligned} e_2 - e_1 &= \Psi^*h_1 - h_1 = \Psi^*(h_0 + e_1) - (h_0 + e_1) \\ &= \mathcal{L}_\zeta h_0 + R_1 \end{aligned}$$

where \mathcal{L} is the Lie derivative, and

$$R_1 := (\Psi^* - \text{Id} - \mathcal{L}_\zeta)h_0 + (\Psi^* - \text{Id})e_1 \quad (3)$$

is a remainder controlled by an expression quadratic in ζ , e_1 and their first derivatives. From an abstract point of view, this control may be expressed in terms of upper bounds for the first derivatives of \exp and the second derivatives of h_0 along the geodesic $t \mapsto \exp_x(t\zeta(x))$, i.e in terms of upper bounds for the Riemann tensor R_0 of g_0 and for $\nabla^2 h_0$ along this geodesic. Similar control applies to the iterated covariant derivatives of R_1 (see Proposition A.1 in the Appendix).

Therefore

$$\mathbb{U}(V, e_2) - \mathbb{U}(V, e_1) = \mathbb{U}(V, \mathcal{L}_\zeta h_0) + R_2 \quad (4)$$

where $R_2 := \mathbb{U}(V, R_1)$ is controlled by an expression quadratic in ζ and e_1 and their derivatives up to the order of Φ (as a differential operator). These terms may be considered as second order error, and will not contribute to the limit $k \rightarrow \infty$ in Definition 2.5 provided the decay conditions are well chosen. This justifies the second assumption of our main result:

Theorem 3.3. *Assume that*

1. $\Phi(h_0)$ is invariant under flows of vector fields,
2. the diffeomorphisms at infinity Ψ_1 and Ψ_2 are such that $\Psi_1^{-1} \circ \Psi_2$ is asymptotic to the identity in the sense of Definition 3.1,
3. the family (S_k) in Definition 2.5 and the section $V \in \mathcal{N}_0$, are such that R_2 defined above satisfies

$$\sup_{S_k} |R_2|_{g_0} \times \text{Vol}_{g_0}(S_k) \xrightarrow{k \rightarrow \infty} 0.$$

Then $m(h, \Psi_1, V) = m(h, \Psi_2, V)$.

Assumption 3 is *ad hoc*. When the curvature of g_0 and its covariant derivatives are bounded, as well as k_0 and its derivatives, it can be replaced by a statement easier to check: see Corollary A.2.

Proof. From Assumption 3, one has

$$m(h, \Psi_1, V) - m(h, \Psi_2, V) = \lim_{k \rightarrow \infty} \oint_{S_k} \mathbb{U}(V, \mathcal{L}_\zeta h_0)(\nu) dS$$

The theorem then follows from the Cancellation Lemma below. The invariance condition on $\Phi(h_0)$ is needed there. \square

Remark 3.4. Invariance of $\Phi(h_0)$ under $\text{Diff}_0(M)$ is equivalent to the fact that $\Phi(h_0)$ is a constant section of a trivial factor—i.e. a factor associated to a trivial representation of the linear group—of the bundle F_0 . The proof of the lemma shows in fact that the pointwise dot products $\langle V, \mathcal{L}_\zeta \Phi(h_0) \rangle_0$, when V varies in \mathcal{N}_0 and ζ among vector fields, are obstructions to $\mathbb{U}(V, \mathcal{L}_\zeta h_0)$ being a divergence. Therefore the invariance condition is presumably almost necessary.

Cancellation Lemma. *Assume that the flows of vector fields leave $\Phi(h_0)$ invariant. Then there exists a differential operator, equivariant under diffeomorphisms,*

$$\mathbb{V} : \Gamma(F_0 \times_{M_0} H_0 \times_{M_0} TM_0) \longrightarrow \Gamma(\Lambda^2 M_0)$$

such that for any $V \in \mathcal{N}_0 = \ker D\Phi_0^$ and any vector field ζ on M_0 one has*

$$\mathbb{U}(V, \mathcal{L}_\zeta h_0) = \text{div}_0 \mathbb{V}(V, h_0, \zeta).$$

Remark 3.5. This lemma is a purely algebraic consequence of the definitions of \mathbb{U} and \mathcal{N}_0 and the $\text{Diff}_0(M_0)$ -invariance of Φ_0 . The non-compactness, decay, etc., assumptions are totally irrelevant here.

Proof. Let ζ be any vector field. We apply Definition 2.1 with $\eta = \mathcal{L}_\zeta h_0$ and $V \in \mathcal{N}_0$:

$$\text{div}_0 \mathbb{U}(V, \mathcal{L}_\zeta h_0) = \langle V, D\Phi_0(\mathcal{L}_\zeta h_0) \rangle_0 = \langle V, \mathcal{L}_\zeta \Phi_0 \rangle_0 = 0$$

using that $D\Phi_0(\mathcal{L}_\zeta h_0) = \mathcal{L}_\zeta \Phi_0$ because of diffeomorphism invariance of the operator Φ , and $\mathcal{L}_\zeta \Phi_0 = 0$ from the assumption.

Assume now that M_0 is orientable. The Hodge star $*_0$ of g_0 conjugates div_0 and the de Rham differential [5, 1.56 and errata], so that $\zeta \mapsto *_0 \mathbb{U}(V, \mathcal{L}_\zeta h_0)$ is an operator:

- whose dependence on the triple (h_0, V, ζ) and on the orientation is equivariant under diffeomorphisms,
- whose values, when h_0 and $V \in \mathcal{N}_0$ are fixed and ζ varies, are closed $(n-1)$ -forms.

From a theorem of Wald [14], there exists a form-valued operator \mathbb{V}^* , equivariant under diffeomorphisms, such that for all ζ , $*_0 \mathbb{U}(V, \mathcal{L}_\zeta h_0) = d\mathbb{V}^*(h_0, V, \zeta)$. The operator $*_0 \mathbb{V}^*$ is the \mathbb{V} claimed in the theorem.

In the non-orientable case, the above reasoning works in the orientation cover, and the formula $\mathbb{U} = \text{div}_0 \mathbb{V}$ there projects down to M_0 (because through the canonical involution of the orientation cover the pulled-back \mathbb{U} , V and h_0 do not change and the Hodge star is changed into its opposite. Following Wald's argument, \mathbb{V}^* is changed into its opposite too, so \mathbb{V} is not). \square

Remark 3.6. Notice that the discussion preceding Theorem 3.3 suggests that, if Φ is of order ℓ , the decay conditions on $\Psi - \text{Id}$ and $\Psi^*h - h_0$ should concern ℓ derivatives of ζ and h . However it is well known that the definition of the ADM mass for example requires only decay of the first derivatives of the asymptotically flat metric, although it comes from the order-2 scalar curvature operator (see Section 4.1.1).

The reason is that we have studied here geometric invariance under a diffeomorphism Ψ acting on h and leaving h_0 fixed. The other point of view in the literature is making Ψ act on h_0 and leave h fixed. These two points of view are conjugated by a diffeomorphism acting on both h and h_0 , and are therefore equivalent. However in the latter, Ψ acts on \mathbb{U} , V , div_0 , and so on, so that the formal study is much more complicated. But with that point of view, control on one less derivative is needed.

To give some detail, in the second point of view one needs to prove that

$$(\Psi^* \mathbb{U})[(\Psi^* V), h - (\Psi^* h_0)] - \mathbb{U}(V, h - h_0)$$

is a divergence up to terms at least quadratic in ζ (of Definition (3.1)) and $h - h_0$. One checks that these terms involve the derivatives of h up to order $\ell - 1$ only.²

²One may notice that the quadratic remainder $Q(V, e)$ that appears in Part 2 of Definition 2.4 involves in general the derivatives of e up to order ℓ . This can be dealt with if Φ is quasilinear of order at least 2, see Remark 4.1.

4 Examples

We recover here known examples, to the invariance of which Theorem 3.3 gives a formal proof.

4.1 The scalar curvature operator

First we take for Φ the scalar curvature operator. Here E, E_0 are the null bundles over respectively M and M_0 , and F, F_0 are the trivial line bundles. Let us write g_0 for the reference metric. We note ∇ its Levi-Civita covariant derivative (without the index 0 to trim notations) and Ric_0 its Ricci tensor. Let $g = g_0 + e$ be another metric on M_0 (at least outside a compact subset). We have

$$\text{Scal}^{g_0+e} = \text{Scal}^{g_0} + \text{DScal}_0(e) + Q(1, e),$$

where the linearization of the scalar curvature at g_0 is, *cf.* Besse [5]:

$$\text{DScal}_0(e) = \text{div}_0(\text{div}_0 e - d(\text{tr } e)) - \langle \text{Ric}_0, e \rangle_0.$$

Here traces are taken with respect to g_0 , and div_0 is the divergence operator of g_0 : for a multi-index J and a tensor T_{iJ} , $\text{div}_0 T_J = \nabla^i T_{iJ}$.

The formal adjoint of DScal_0 is

$$\text{DScal}_0^*(V) = \nabla^2 V + \Delta_0 V g_0 - V \text{Ric}_0$$

where $\Delta_0 = -\nabla^i \nabla_i$ is the geometric Laplacian. The equation $\text{DScal}_0^* V = 0$ is equivalent to the fact that the metrics $g_0 \pm V^2 dt^2$ on $M_0 \times \mathbb{R}$ are Einstein, *cf.* [8].

The charge boundary integrand computed with Definition 2.1 is:

$$\mathcal{U}(V, e) = V(\text{div}_0 e - d(\text{tr } e)) - \iota_{\nabla V} e + (\text{tr } e) dV \quad (5)$$

where ι_X denotes the contraction of a vector X with a (covariant) tensor. Up to second-order terms this is the formula of [8], see also [11].

One checks also that, if the operator norm of $g_0^{-1}e$ is not greater than $1/2$ (so that, with the help of the Neumann series, $|g^{-1} - g_0^{-1}|_0 \leq 2|e|_0$ and $|\nabla g^{-1}|_0 \leq 4|\nabla e|_0$), the quadratic remainder of Equation (2) is bounded:

$$Q(1, e) \leq C(|\nabla e|_0^2 + |e|_0 |\nabla^2 e|_0) \quad (6)$$

where C is a dimensional constant (independent of g_0 in particular).

Remark 4.1. This suggests that the appropriate decay conditions for the definitions of masses and center of mass—to be presented below—should concern two derivatives of $g - g_0$. But it is well known that only control on the first derivatives are needed. This comes from the fact that, using an integration by part whenever a second derivative of e occurs, one may write

$$Q(V, e) = VQ(1, e) = Q_1(V, e) + Q'_1(\nabla V, e) + \text{div}_0 Q_2(V, e),$$

where Q_1, Q'_1 and Q_2 are linear in their first argument, and at least quadratic in e and its first derivatives. Thus in Definition 2.4, integrability of $Q(V, e)$ may be replaced by decay of e and ∇e such that Q_1 and Q'_1 are integrable and integrals of Q_2 over large spheres vanish in the limit.

4.1.1 The asymptotically flat case

First we take $M_0 = \mathbb{R}^n$ and g_0 the canonical flat metric.

This background space is asymptotically rigid in the sense of Remark 3.2, see [6] and [4]. Namely, any two systems of coordinates at infinity on (M, g) of class C^2 , such that in both, the coefficients of g satisfy:

$$e_{ij} := g_{ij} - \delta_{ij} = O(r^{-\tau}), \quad \partial_k g_{ij} = O(r^{-\tau-1}) \quad \text{with } \tau > 0, \quad (7)$$

(with $r = (\sum_i x_i^2)^{1/2}$), differ in a diffeomorphism at infinity $\Psi \circ A$, where A is a Poincaré transform $A: x \mapsto Rx + T$, $R \in O(n)$, $T \in \mathbb{R}^n$, and Ψ is asymptotic to the identity in the sense of Definition 3.1, with ζ there satisfying

$$\zeta^i = O(r^{1-\tau}), \quad \partial_j \zeta^i = O(r^{-\tau}), \quad \partial_j \partial_k \zeta^i = O(r^{-\tau-1}) \quad (8)$$

(here of course $\exp_x \zeta(x)$ has coordinates $x^i + \zeta^i(x)$).

The space \mathcal{N}_0 consists of affine functions, which grow like r at infinity. Part 1 of Definition 2.4 is satisfied if $r \text{Scal}^g$ is integrable. Part 2 of that definition is satisfied if (7) and $\partial_k \partial_l g_{ij} = O(r^{-\tau-2})$ hold with $\tau > \frac{n-1}{2}$, since, because of (6), the quadratic term $Q(V, e)$ is then $O(r^{-2\tau-1})$, with $-2\tau - 1 < -n$. As explained in Remark 4.1, the control of the second derivatives of g is in fact superfluous.

Because of the asymptotic rigidity stated above, if we can apply Theorem 3.3 to a vector field ζ satisfying (8) for appropriate τ , we will obtain an actual invariance of the total-charge linear form on \mathcal{N}_0 . We could apply Corollary A.2, but it is not optimal for the usual asymptotic flatness assumptions (7). Let us instead compute in coordinates for example the last term of R_1 in (3):

$$\begin{aligned} \zeta^c(x) \int_0^1 \partial_c e_{ij}(x + t\zeta) dt + e_{aj}(x + \zeta) \partial_i \zeta^a(x) + e_{ib}(x + \zeta) \partial_j \zeta^b(x) \\ + e_{ab}(x + \zeta) \partial_i \zeta^a(x) \partial_j \zeta^b(x), \end{aligned} \quad (9)$$

which is $O(r^{-2\tau})$ if (7) and (8) hold, and whose derivatives are $O(r^{-1-2\tau})$ if moreover $\partial_k \partial_l g_{ij} = O(r^{-\tau-2})$. The same estimates apply to the first term of the right-hand side of (3), which is here $\sum_k \partial_i \zeta^k \partial_j \zeta^k$. So the term R_2 of Equation (4) is $O(r^{-2\tau})$ under these assumptions. If $2\tau > n - 1$, its integral over large coordinates spheres vanishes in the limit. This is what is needed to apply Theorem 3.3. (Again, in view of Remark 3.6, the control of the second derivatives of g is in fact superfluous.)

Therefore, under these decay assumptions we recover the well-known ADM mass:

$$\mathbb{U}_j(1, e) = \sum_i \partial_i e_{ij} - \partial_j e_{ii}$$

and center of mass:

$$\mathbb{U}_j(x^a, e) = \sum_i [x^a (\partial_i e_{ij} - \partial_j e_{ii}) - e_{ai} + e_{ii} \delta_{aj}].$$

Of course the subspace \mathcal{N}'_0 of constant functions is invariant under the isometries of \mathbb{R}^n . Since constant functions grow slower at infinity than general affine functions, this allows to relax the asymptotic decay conditions to define the ADM mass: as is well known, its definition only requires Scal^g to be integrable and $\tau > \frac{n-2}{2}$ in (7).

4.1.2 The asymptotically hyperbolic case

Here we set $M_0 = \mathbb{H}^n$ endowed with the hyperbolic metric $g_0 = dr^2 + (\sinh^2 r) \check{g}$, where \check{g} is the canonical metric of the unit sphere \mathbb{S}^{n-1} and r the distance to a fixed point. We refer to [10] and [8] for a detailed treatment, which includes more general, similar cases (notice that their coordinate r is the hyperbolic sine of ours).

Asymptotic rigidity holds here [10, Theorem 3.3], [8, proof of Theorem 2.3], as mentioned in Remark 3.2: if a metric g , pulled-back on \mathbb{H}^n via two diffeomorphisms Ψ_q , $1 \leq q \leq 2$, satisfies

$$|\Psi_q^* g - g_0|_0 = O(e^{-\tau r}) \quad \text{and} \quad |\nabla(\Psi_q^* g)|_0 = O(e^{-\tau r}) \quad (10)$$

for some $\tau > 1$, then $\Psi_1^{-1} \circ \Psi_2 = \Psi \circ A$, where A is an isometry of g_0 , and Ψ is asymptotic to the identity in the sense of Definition 3.1, with

$$|\zeta|_0, |\nabla \zeta|_0, |\nabla^2 \zeta|_0 = O(e^{-\tau r}). \quad (11)$$

We set $e := \Psi_1^* g - g_0$.

The space \mathcal{N}_0 is generated by the coordinates of the canonical isometric embedding

$$(V_{(0)}, V_{(1)}, \dots, V_{(n)}) : \mathbb{H}^n \rightarrow \mathbb{R}^{1,n}$$

into the Minkowski space. In spherical coordinates one has $V_{(0)} = \cosh r$ and $V_{(i)} = (\sinh r) \xi^i$, $i \geq 1$, where $(\xi^1, \dots, \xi^n) : \mathbb{S}^{n-1} \rightarrow \mathbb{R}^n$ is the canonical embedding. These functions and their g_0 -gradient are $O(e^r)$. So Part 1 of Definition 2.4 is satisfied if $e^r \text{Scal}^g$ is integrable. Part 2 of that definition is satisfied if (10) and $|\nabla^2 g|_0 = O(e^{-\tau r})$ hold with $\tau > \frac{n}{2}$, since then

$$Q(V, e) d\text{vol}_0 = Q(V, e) (\sinh^{n-1} r) dr d\text{vol}_{\check{g}} = O(e^{(n-2\tau)r}) dr d\text{vol}_{\check{g}}$$

because of (6), and $n - 2\tau < 0$. (As explained in Remark 4.1 the control of the second derivatives of g is in fact superfluous.)

As for changes of asymptotically hyperbolic coordinates, because of asymptotic rigidity, Theorem 3.3 applied to ζ 's satisfying (11) (with appropriate τ) will yield an actual invariance of the total-charge form on \mathcal{N}_0 . We need to check Assumption 3: we use Corollary A.2. We take for S_r the distance spheres. Their volume is $|S_r| = \sinh^{n-1} r$. We have $|V| + |\nabla V| = O(e^r)$, and $|\mathbb{U}(V, e)| \leq U(|V| + |\nabla V|)(|e| + |\nabla e|)$ for some constant U . Hence the three conditions of Corollary A.2 are satisfied when (10), (11) and $|\nabla^2 g|_0 = O(e^{-\tau r})$ hold with $\tau > \frac{n}{2}$. (Again, in view of Remark 3.6, the control of the second derivatives of g is in fact superfluous.)

Thus, under the condition $\tau > \frac{n}{2}$ in (10), we recover here the Chruściel-Herzlich momentum in the asymptotically hyperbolic setting, given by the limit of the integrals of (5) over large r -spheres, when V is one of the $V_{(\mu)}$ above. These $V_{(\mu)}$ are reshuffled by a Lorentz matrix through the action of the isometries of g_0 . Therefore so are the associated total charges.

4.2 The operator of constraints of general relativity

Here we take for Φ the constraints on Cauchy initial data in General Relativity. The geometric data are a couple (g, k) of a first and a second fundamental form of a hypersurface in an ambient space-time, and

$$\Phi : \Gamma(\mathcal{M} \times_M S_2 M) \longrightarrow \Gamma(\mathbb{R} \oplus T^* M)$$

$$(g, k) \longmapsto \begin{pmatrix} \text{Scal}^g + (\text{tr}_g k)^2 - |k|_g^2 \\ 2(\text{div}_g k - d(\text{tr}_g k)) \end{pmatrix} =: \begin{pmatrix} \Phi^H(g, k) \\ \Phi^M(g, k) \end{pmatrix}.$$

The invariance condition for Φ_0 imposes Φ_0^M to vanish and Φ_0^H to be a constant 2Λ : these are the constraint equations with cosmological constant in vacuum for (g_0, k_0) . The test-section V is a couple (f, α) of a function and a 1-form over M_0 . The linearized operators at (g_0, k_0) , for a variation $e = (\dot{g}, \dot{k})$, are

$$\begin{aligned} D\Phi_0^H(e) &= \text{div div } \dot{g} + \Delta \text{tr } \dot{g} - \langle \text{Ric}_0 - 2k_0 \circ k_0 + 2(\text{tr } k_0)k_0, \dot{g} \rangle \\ &\quad - 2\langle k_0, \dot{k} \rangle + 2\text{tr } k_0 \text{tr } \dot{k} \\ D\Phi_0^M(e) &= \iota_{\nabla \text{tr } \dot{g}} k_0 - k_0^{ij} \nabla \dot{g}_{ij} - 2(\text{div}(\dot{g} \circ k_0) - d\langle k_0, \dot{g} \rangle) \\ &\quad + 2(\text{div } \dot{k} - d(\text{tr } \dot{k})) \end{aligned}$$

where Ric_0 is the Ricci tensor of g_0 ; traces, divergences, index lowering and raising (implicit when needed), *etc.* are taken with respect to g_0 ; and \circ is the composition of 2-tensors (using g_0): $(A \circ B)_{ij} := g_0^{kl} A_{ik} B_{lj}$. From there one computes (beware of the sign convention: here it is Riemannian)

$$\begin{aligned} \langle V, D\Phi_0(e) \rangle &= f D\Phi_0^H(e) + \langle \alpha, D\Phi_0^M(e) \rangle \\ &= \text{div } \mathbb{U}(V, e) + \langle (\partial_g \Phi)_0^*(V), \dot{g} \rangle + \langle (\partial_k \Phi)_0^*(V), \dot{k} \rangle \end{aligned}$$

with

$$\begin{aligned} (\partial_g \Phi)_0^*(f, \alpha) &= \nabla^2 f + (\Delta f)g_0 - f(\text{Ric}_0 - 2k_0 \circ k_0 + 2(\text{tr } k_0)k_0) \\ &\quad + \mathcal{L}_\alpha k_0 - (\text{div } \alpha)k_0 - (\langle \nabla \alpha, k_0 \rangle + \langle \alpha, \text{div } k_0 \rangle)g_0 \\ (\partial_k \Phi)_0^*(f, \alpha) &= -2f(k_0 - \text{tr } k_0 g_0) - \mathcal{L}_\alpha g_0 + 2(\text{div } \alpha)g_0 \end{aligned}$$

and

$$\begin{aligned} \mathbb{U}(V, e) &= f(\text{div } \dot{g} - d(\text{tr } \dot{g})) - \iota_{\nabla f} \dot{g} + (\text{tr } \dot{g})df \\ &\quad + 2(\iota_\alpha \dot{k} - (\text{tr } \dot{k})\alpha) + (\text{tr } \dot{g})\iota_\alpha k_0 + \langle k_0, \dot{g} \rangle \alpha - 2\iota_\alpha(\dot{g} \circ k_0). \end{aligned}$$

Up to terms quadratic in \dot{g} and \dot{k} , this is the general formula of Chruściel, Jezierski and Łęski [9, §2] (notice that their Y is $-g_0^{-1}\alpha$ here, and their P is not in fully covariant form, whereas k here is). See also [12] (beware of the sign conventions for the momentum constraints and the divergence operator).

One sees also that the quadratic error $Q(V, e)$ of (2), that appears in Definition 2.4, is bounded, up to a multiplicative constant independent of g_0 and k_0 , by

$$|f| \left(|\dot{g}|^2 |k_0|^2 + |\nabla \dot{g}|^2 + |\dot{g}| |\nabla^2 \dot{g}| + |\dot{k}|^2 \right) + |\alpha| \left(|\dot{g}|^2 (|k_0|^2 + |\nabla k_0|) + |\nabla \dot{g}|^2 + |\dot{k}|^2 + |\dot{g}| |\nabla \dot{k}| \right) \quad (12)$$

when the operator norm of $g_0^{-1} \dot{g}$ is no more than $1/2$. (As in Remark 4.1, the assumption of integrability of $Q(V, e)$ may be replaced by decay conditions on e that involve neither $\nabla^2 \dot{g}$ nor $\nabla \dot{k}$.)

It is known [13] that the kernel \mathcal{N}_0 of the operator $D\Phi_0^*$ contains exactly the couples (f, α) such that $(f, g_0^{-1} \alpha)$ is the normal-tangential decomposition of the restriction along M_0 of a Killing vector field of the (Lorentzian) Choquet-Bruhat development of (M_0, g_0, k_0) .

If one wants an asymptotic rigidity result as mentioned in Remark 3.2 for a couple (g_0, k_0) , one may consider g_0 as in the examples of Section 4.1 (flat or hyperbolic), and $k_0 = \lambda_0 g_0$. The invariance condition on $\Phi(g_0, k_0)$ imposes λ_0 to be constant, so that isometries of g_0 also fix k_0 . The boundary integrand then reduces to

$$\mathbb{U}(V, e) = (f \operatorname{div}_0 - \iota_{\nabla f})(\dot{g} - (\operatorname{tr} \dot{g})g_0) + 2\iota_\alpha(\dot{k} - \lambda_0 \dot{g} - \operatorname{tr}(\dot{k} - \lambda_0 \dot{g})g_0).$$

This is the Chruściel-Jeziński-Łęski expression for the Bondi mass [9, §3], under asymptotic decay conditions that do however not allow gravitational radiation. When $\lambda_0 = 0$, \mathbb{U} decomposes into a (f, \dot{g}) -part equal to the \mathbb{U} of Section 4.1, and a (α, \dot{k}) -part: $2\iota_{g_0^{-1}\alpha}(\dot{k} - \operatorname{tr}_0 \dot{k} g_0)$, that give respectively the usual ADM and Abbott-Deser momenta.

A Explicit control of the remainder R_1

Recall that we are given the data $h_0 = (g_0, k_0)$ of a Riemannian metric g_0 and a natural tensor k_0 over M_0 . We assume that g_0 is complete. We consider a diffeomorphism $\Psi : x \mapsto \exp_x \zeta(x)$ (where \exp is the g_0 exponential map), and search for estimates of $\nabla^\ell(\Psi^* k - k)$ and $\nabla^\ell(\Psi^* k - k - \mathcal{L}_\zeta k)$, when k is a tensor field.

We first introduce some notations. Let $|k|_\ell(x) := |k(x)|_0 + \dots + |\nabla^\ell k(x)|_0$, and $\|k\|_\ell(x)$ be the supremum of $|k|_\ell$ along the geodesic $t \mapsto \exp_x t\zeta(x)$, $t \in [0, 1]$.

Proposition A.1. *1. There exists a universal constant $\varepsilon > 0$ such that, if ζ and the sectional curvatures κ_0 of g_0 satisfy $\kappa_0 |\zeta|_0^2 \leq \varepsilon$ and $|\nabla \zeta| \leq \varepsilon$ on M_0 , then $\Psi := \exp \circ \zeta$ is a diffeomorphism such that*

$$\frac{1}{4}g_0 \leq \Psi^* g_0 \leq 4g_0.$$

2. Assume that Point 1 holds. Assume that the Riemann tensor R_0 of g_0 , ζ and their covariant derivatives up to order ℓ are bounded. Then there exists a constant C such that the term R_1 in Equation (3) satisfies

$$|\nabla^\ell R_1|_0 \leq C |\zeta|_{\ell+1} (|\zeta|_{\ell+1} \|k_0\|_{\ell+2} + \|e_1\|_{\ell+1}).$$

Proof of Proposition A.1. Since norms, covariant derivatives and curvatures always refer to g_0 in this proof, we shall omit the indices 0. The notation $|\cdot|$ will here denote the pointwise g_0 -norm of its argument, and $\|\cdot\|$ its supremum along the geodesic $t \mapsto \exp_x t\zeta(x)$. The letter C will refer to a positive constant that may change from line to line.

Proof of Part 1. Let us denote $\Psi_t(x) := \exp_x t\zeta(x)$. We fix $x \in M_0$, an orthonormal frame E_i of $T_x M_0$, and a vector $X \in T_x M_0$, and write $\gamma(t) := \Psi_t(x)$ for $t \in [0, 1]$. Let $T(t) := \frac{\partial \gamma}{\partial t}$. For short we will write ∇_T instead of $\gamma^*(\nabla)_{\partial_t}$ or $\frac{\nabla}{dt}$. Let $E_i(t)$ be the parallel transport of E_i along γ . When k is a section of a tensor bundle K , we denote by $k(t)$ the value of $k(\gamma(t))$ in the trivialization of $\gamma^* K$ given by the frame $E_i(t)$. Note that $k'(t) = (\nabla_T k)(t)$.

Let $X(t) := T_x \Psi_t X$. It is the Jacobi field along γ with initial conditions $X(0) = X$, $X'(0) = \nabla_X \zeta(x)$. Classical comparison techniques insure that, provided with sufficiently small upper bounds for

$|T(t)|^2 \kappa(\gamma(t))$ (where κ is the pointwise supremum of the sectional curvatures) and $|\nabla\zeta(x)|$, one has $1/2 |X(0)| \leq |X(t)| \leq 2 |X(0)|$ along γ (which implies that Ψ is a diffeomorphism). Such upper bounds hold if $\kappa |\zeta|^2$ and $|\nabla\zeta|$ are bounded by a sufficiently small $\varepsilon > 0$ everywhere on M_0 . This gives Part 1 of the proposition. From now on we assume that it holds.

Proof of Part 2 when $\ell = 0$. Let $P(t)$ be the matrix of $T_x \Psi_t$ with respect to the frames $E_i(0)$ and $E_i(t)$. It is bounded in a fixed neighborhood of the identity matrix. From the Jacobi equation $|P''(t)| \leq C \|R\| (x) |\zeta|^2 (x)$ and therefore, for $t \leq 1$, because of the initial conditions:

$$|P'(t)| \leq C (\|R\| (x) |\zeta|^2 (x) + |\nabla\zeta| (x)).$$

For a tensor field k , $\Psi_t^* k(x)$ has value $P(t)^{-1} \cdot k(t)$ in the frame $E_i(0)$ (the action of GL_n defining the tensor bundle in which k lives is understood). Thus $1/C |k| \leq |\Psi^* k| \leq C |k|$ for some positive constant C ;

$$\begin{aligned} |\Psi^* k(x) - k(x)| &\leq \int_0^1 \left| \frac{d}{dt} (P(t)^{-1} \cdot k(t)) \right| dt \leq C \left[(\|R\| |\zeta|^2 + |\nabla\zeta|) \|k\| + |\zeta| \|\nabla k\| \right] (x) \\ &\leq C |\zeta|_1 (x) \|k\|_1 (x); \end{aligned} \quad (13)$$

when R and ζ are bounded; and with the help of a Taylor formula

$$\begin{aligned} |\Psi^* k(x) - k(x) - \mathcal{L}_\zeta k(x)| &\leq \int_0^1 (1-t) \left| \frac{d^2}{dt^2} (P(t)^{-1} \cdot k(t)) \right| dt \\ &\leq C \left[|\zeta|^2 \|\nabla^2 k\| + |\zeta| (\|R\| |\zeta|^2 + |\nabla\zeta|) \|\nabla k\| + \|R\| |\zeta|^2 \|k\| \right] (x) \\ &\leq C |\zeta|_1^2 (x) \|k\|_2 (x). \end{aligned} \quad (14)$$

Plugging Equations (13) and (14) into (3) with respectively $k = e_1$ and $k = k_0$ gives Part 2 of the proposition in case $\ell = 0$.

Proof of Part 2 when $\ell = 1$. The proof is an induction on ℓ . We present the first step in detail. We parallel-transport the previous construction along a geodesic through x : let us set

- $X, Y \in T_x M_0$,
- $\gamma_1(s) := \exp_x(sX)$,
- $X(s, 0), Y(s, 0)$ the parallel transports of X and Y respectively along γ_1 ,
- $\gamma(s, t) := \Psi_t(\gamma_1(s))$,
- $X(s, t) := T_{\gamma_1(s)} \Psi_t X(s, 0) = \frac{\partial \gamma}{\partial s}$ the Jacobi field along $t \mapsto \gamma(s, t)$ with initial data $X(s, 0)$ and $\nabla_{X(s, 0)} \zeta(\gamma_1(s))$, and $Y(s, t) := T_{\gamma_1(s)} \Psi_t Y(s, 0)$ similarly,
- $E_i(t)$ the parallel transport of an orthonormal frame at x along $t \mapsto \exp_x t\zeta(x)$.

We write $T(t, s) := \frac{\partial \gamma}{\partial t}(s, t)$, and $\nabla_T = \frac{\nabla}{\partial t}$, $\nabla_X = \frac{\nabla}{\partial s}$ for short. The covariant derivative of the Jacobi equation satisfied by Y in the direction of X may be written

$$\begin{aligned} \nabla_T \nabla_T (\nabla_X Y) - R_{T, \nabla_X Y} T &= (\nabla_X R)_{T, Y} T + R_{\nabla_T X, Y} T + R_{T, Y} \nabla_T X \\ &\quad + R_{T, X} \nabla_T Y - \nabla_T (R_{T, X} Y). \end{aligned} \quad (15)$$

This is a second order ODE for $\nabla_X Y$, whose homogeneous part is the Jacobi equation (a classical fact in ODE theory); the initial conditions are

$$\begin{aligned} \nabla_X Y(s, 0) &= 0, \\ \frac{\nabla}{\partial t} \nabla_X Y(s, 0) &= R_{T, X} Y(s, 0) + \frac{\nabla}{\partial s} \frac{\nabla}{\partial t} Y(s, 0) = R_{T, Y} X(s, 0) + (\nabla^2 \zeta)_{X, Y}(s, 0). \end{aligned}$$

If R , ∇R and ζ are bounded, the source term in the ODE (15) is bounded by

$$C |\zeta|_1(x) |X(0,0)| |Y(0,0)|$$

because of the part $\ell = 0$ of the proof applied to $X(s, t)$ and $Y(s, t)$. Moreover

$$|\nabla_T \nabla_X Y|_{t=0} \leq C(|\zeta| + |\nabla^2 \zeta|) |X(0,0)| |Y(0,0)|.$$

The method of variation of parameters then shows that

$$|\nabla_X Y(s, t)| \leq C |\zeta|_2(x), \quad |\nabla_T \nabla_X Y(s, t)| \leq C |\zeta|_2(x)$$

Let $\Gamma(t)$ be the matrix of the linear map $X(0,0) \otimes Y(0,0) \mapsto \nabla_X Y(0, t)$ with respect to the frames $E_i(0)$ and $E_i(t)$. Then $\Psi_t^* \nabla - \nabla$ has matrix $P(t)^{-1} \Gamma(t)$ in the frame $E_i(0)$ ($P(t)$ being again the matrix of $T_x \Psi_t$). From the previous paragraph, the part $\ell = 0$ of the proof and the ODE (15) for $\Gamma''(t)$, one has

$$|P(t)^{-1} \Gamma(t)| \leq C |\zeta|_2, \quad \left| \frac{d}{dt} P(t)^{-1} \Gamma(t) \right| \leq C |\zeta|_2, \quad \left| \frac{d^2}{dt^2} P(t)^{-1} \Gamma(t) \right| \leq C |\zeta|_2^2.$$

Therefore as in the case $\ell = 0$,

$$|\Psi^* \nabla - \nabla| \leq C |\zeta|_2, \quad |\Psi^* \nabla - \nabla - \mathcal{L}_\zeta(\nabla)| \leq C |\zeta|_2^2.$$

Thus for any tensor field k :

$$|\nabla(\Psi^* k - k)| = |(\Psi^* - \text{Id})(\nabla k) - (\Psi^* \nabla - \nabla) \Psi^* k| \leq C |\zeta|_2 \|k\|_2 \quad (16)$$

$$\begin{aligned} |\nabla(\Psi^* k - k - \mathcal{L}_\zeta k)| &= |(\Psi^* - \text{Id} - \mathcal{L}_\zeta) \nabla k - (\Psi^* \nabla - \nabla)(\Psi^* k - k) \\ &\quad - (\Psi^* \nabla - \nabla - \mathcal{L}_\zeta(\nabla)) k| \\ &\leq C |\zeta|_2^2 \|k\|_2 \end{aligned} \quad (17)$$

(where the result for $\ell = 0$ has been used as well). The case $\ell = 1$ of the proposition follows from (16) with $k = e_1$ and from (17) with $k = k_0$ in the ∇ -derivative of Equation (3).

The induction on ℓ uses a similar construction and a similar ODE for iterated derivatives $\nabla_{X_1} \cdots \nabla_{X_\ell} Y$. They result in estimates

$$|\nabla^{\ell-1}(\Psi^* \nabla - \nabla)| \leq C |\zeta|_{\ell+1}, \quad |\nabla^{\ell-1}(\Psi^* \nabla - \nabla - \mathcal{L}_\zeta \nabla)| \leq C |\zeta|_{\ell+1}^2$$

when R , ζ and their derivatives up to order ℓ are bounded (which allows to easily control the otherwise complicated source term and initial conditions of the ODE). The claimed control on $\nabla^\ell R_1$ is deduced as above. The induction is straightforward but lengthy, so we do not give the details. \square

We use Proposition A.1 to replace Assumption 3 of Theorem 3.3 with a less *ad hoc* statement. Let ℓ be the order of the differential operator Φ . Recalling that $\mathbb{U}(V, \eta)$ is a linear differential operator of order $\ell - 1$ in V and η , there exists a nonnegative continuous function U such that

$$\forall x \in M_0, |\mathbb{U}(V, \eta)| \leq U(x) |V|_{l-1}(x) |\eta|_{l-1}(x). \quad (18)$$

Noticing that the operator \mathbb{U} is moreover a differential operator of order at most ℓ in the reference data (g_0, k_0) , equivariant under diffeomorphisms, the function U in Equation (18) is bounded when the curvature of g_0 , its derivatives up to order $\ell - 2$ and the derivatives of k_0 up to order ℓ are bounded. Applying (18) to $\eta = \mathcal{L}_\zeta h_0$, we then obtain the following direct consequence of Proposition A.1:

Corollary A.2. *Assume that Assumption 2 of Theorem 3.3 holds. Assume moreover that the Riemann tensor of g_0 and its derivatives up to order $\ell - 1$, and k_0 and its derivatives up to order $\ell + 1$ are bounded. Then Assumption 3 of Theorem 3.3 is implied by the following conditions on the family of hypersurfaces S_k , ζ (of Definition 3.1) and e_1 :*

$$\begin{aligned} &|\zeta|_{\ell-1} \text{ is bounded,} \\ &\text{Vol}(S_k) \sup_{S_k} (|V|_{\ell-1} |\zeta|_\ell^2) \xrightarrow{k \rightarrow \infty} 0, \\ &\text{Vol}(S_k) \sup_{S_k} (|V|_{\ell-1} \|e_1\|_\ell^2) \xrightarrow{k \rightarrow \infty} 0. \end{aligned}$$

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